

# The $\mathcal{L}$ -invariant, the dual $\mathcal{L}$ -invariant, and families

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*Dedicated to Glenn Stevens on the occasion of his 60th birthday*

## Abstract

Given a rank two trianguline family of  $(\varphi, \Gamma)$ -modules having a noncrystalline semistable member, we compute the Fontaine–Mazur  $\mathcal{L}$ -invariant of that member in terms of the logarithmic derivative, with respect to the Sen weight, of the value at  $p$  of the trianguline parameter. This generalizes prior work, in the case of Galois representations, due to Greenberg–Stevens and Colmez.

## §0 Introduction

In the remarkable paper [GS], Greenberg and Stevens proved a formula, conjectured by Mazur, Tate, and Teitelbaum in [MTT], for the derivative at  $s = 1$  of the  $p$ -adic  $L$ -function of an elliptic curve  $E/\mathbf{Q}$  when  $p$  is a prime of split multiplicative reduction. The novel quantity in this formula was the so-called  $\mathcal{L}$ -invariant, namely  $\mathcal{L}(E) = \log_p(q_E) / \text{ord}_p(q_E)$  where  $q_E \in p\mathbf{Z}_p$  generates the kernel of the Tate uniformization  $\mathbf{G}_{m, \mathbf{Q}_p}^{\text{an}} \rightarrow E_{\mathbf{Q}_p}^{\text{an}}$ . The proof of the Greenberg–Stevens Theorem had two main steps. On the one hand, a two-variable  $p$ -adic  $L$ -function was constructed, allowing the sought derivative to be computed in terms of the derivative of the Hecke  $U_p$ -operator with respect to the weight. On the other hand, a local Galois cohomology computation was used to relate this derivative of  $U_p$  to the  $\mathcal{L}$ -invariant, as in the formula [GS, (0.15)].

This short paper extends [GS, (0.15)], by extending the technique of its proof, to  $(\varphi, \Gamma)$ -modules over the Robba ring. The main result, which is proved in §2.3, is as follows.

**Theorem.** *Let  $X$  be an analytic space over a finite extension  $E$  of  $\mathbf{Q}_p$ , let  $\delta, \eta: \mathbf{Q}_p^\times \rightarrow \mathcal{O}_X^\times$  be continuous characters, and let the  $(\varphi, \Gamma)$ -module  $D$  over  $\mathcal{R}_X$  be an extension of  $\mathcal{R}_X(\delta)$  by  $\mathcal{R}_X(\eta)$ . Assume that  $P \in X$  is such that the specialization  $D_0$  of  $D$  at  $P$  is, up to twist, noncrystalline semistable. Then the differential form*

$$d \log(\eta \delta^{-1}(p)) - \mathcal{L}(D_0) \cdot d \text{wt}(\eta \delta^{-1}) \in \Omega_{X/E}$$

*vanishes at  $P$ .*

We hope the extension might motivate the generalization of other techniques of [GS] to eigenvarieties whose triangulations possess a two-step graded piece that is noncrystalline semistable up to twist, and not necessarily étale. Although we had originally hoped to study the Hodge–Tate property in a uniform manner by formulating “dual  $\mathcal{L}$ -invariants”, we show at the end of §2.3 that such a technique necessarily cannot work.

There is a rich history of notions of  $\mathcal{L}$ -invariant, especially for modular forms of higher weight, and comparisons among them, for which we refer to [C]. Especially, the formula [GS, (0.15)] was previously generalized to the nonordinary setting by Colmez [C2, Théorème 0.5], using a study of the adjoint representation and delicate computations in Fontaine’s rings. Readers who are comfortable translating between the languages of crystalline periods and triangulations will find that his result is none other than the special case of our result where the rank two family of  $(\varphi, \Gamma)$ -modules arises from a family of Galois representations. (For other generalizations see [B], which we learned of after the writing of this paper.)

For many years, Glenn Stevens has provided me with inspiration and support. It is a pleasure to dedicate this article to him.

## §1 $\mathcal{L}$ -invariants in arithmetic

### §1.1 Abstract $\mathcal{L}$ -invariants

Let  $E$  be a field,  $X$  a two-dimensional  $E$ -vector space with dual  $X^* = \text{Hom}_E(X, E)$ , and  $\lambda^*, \mu^* \in X^*$  a distinguished ordered basis. For  $x^* = a\lambda^* + b\mu^* \in X^*$ , we write  $\langle \lambda^*, x^* \rangle = a$  and  $\langle \mu^*, x^* \rangle = b$ .

The  $\mathcal{L}$ -invariant is the bijection

$$\mathcal{L}: \mathbf{P}(X) \xrightarrow{\sim} \mathbf{P}^1(E), \quad Ex \mapsto (\lambda^*(x) : \mu^*(x)),$$

and the *dual  $\mathcal{L}$ -invariant* is the bijection

$$\mathcal{L}^*: \mathbf{P}(X^*) \xrightarrow{\sim} \mathbf{P}^1(E), \quad Ex^* \mapsto (-\langle \mu^*, x^* \rangle : \langle \lambda^*, x^* \rangle).$$

We often confuse  $\mathcal{L}$  (resp.  $\mathcal{L}^*$ ) with its composition with the projection  $X \setminus \{0\} \rightarrow \mathbf{P}(X)$  (resp.  $X^* \setminus \{0\} \rightarrow \mathbf{P}(X^*)$ ).

Note that two lines  $L \subset X$  and  $L^* \subset X^*$  are orthogonal if and only if  $\mathcal{L}(L) = \mathcal{L}^*(L^*)$ .

### §1.2 Arithmetic setup

We fix a prime  $p$ , choose an algebraic closure  $\mathbf{Q}_p^{\text{alg}}$  of  $\mathbf{Q}_p$ , and let  $G_{\mathbf{Q}_p} = \text{Gal}(\mathbf{Q}_p^{\text{alg}}/\mathbf{Q}_p)$ . The base field (not to be confused with the coefficient field) of all Galois representations (resp.  $(\varphi, \Gamma)$ -modules) will always be  $\mathbf{Q}_p$  in this paper, so we omit it from the notations of continuous Galois cohomology (resp. Herr cohomology).

Write  $\mathbf{Q}_p^{\times, \wedge} = \varprojlim_n \mathbf{Q}_p^{\times} / (\mathbf{Q}_p^{\times})^{p^n}$  for the pro- $p$  completion of  $\mathbf{Q}_p^{\times}$ . Kummer theory gives rise to an identification  $\mathbf{Q}_p^{\times, \wedge} \xrightarrow{\sim} H^1(\mathbf{Z}_p(1))$ , and using flat base change for  $\otimes_{\mathbf{Z}_p} S$  we deduce from this an isomorphism  $\mathbf{Q}_p^{\times, \wedge} \otimes_{\mathbf{Z}_p} S \xrightarrow{\sim} H^1(S(1))$ , whose inverse we denote by  $q$ . On the other hand, the local Artin map induces an isomorphism  $\mathbf{Q}_p^{\times, \wedge} \xrightarrow{\sim} G_{\mathbf{Q}_p}^{p\text{-ab}}$  (the maximal pro- $p$ -abelian quotient), and by composition an identification  $H^1(S) = \text{Hom}_{\mathbf{Z}_p}(G_{\mathbf{Q}_p}^{p\text{-ab}}, S) = \text{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p^{\times, \wedge}, S)$ , which we denote by  $q^*$ . The pairing

$$H^1(S(1)) \times H^1(S) \cong (\mathbf{Q}_p^{\times, \wedge} \otimes_{\mathbf{Z}_p} S) \times \text{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p^{\times, \wedge}, S) \rightarrow S,$$

given  $q \times q^*$  followed by evaluation of homomorphisms, coincides (up to sign) with the Tate pairing, given by cup product and the local invariant map. It is therefore perfect when  $S$  is a finite field extension of  $\mathbf{Q}_p$ .

We denote by  $\log_p: \mathbf{Q}_p^{\times} \rightarrow \mathbf{Q}_p$  the natural extension of the logarithm satisfying  $\log_p(p) = 0$ , and by  $\text{ord}_p: \mathbf{Q}_p^{\times} \rightarrow \mathbf{Z}$  the  $p$ -adic valuation; each of these induces by continuity a homomorphism  $\mathbf{Q}_p^{\times, \wedge} \rightarrow \mathbf{Q}_p$ , which we denote by the same name. Under  $q^*$ , the  $S$ -module  $H^1(S)$  is free of rank two on the ordered basis  $\log_p, \text{ord}_p$ .

The statement of the theorem assumes the reader is familiar with the language of  $(\varphi, \Gamma)$ -modules  $D$  over Berger's Robba ring  $\mathcal{R}_X$  with coefficients in an analytic space  $X$  over  $\mathbf{Q}_p$ . However, in our work below we will only need the case where  $X$  is the spectrum of a finite  $\mathbf{Q}_p$ -algebra  $B$ , in which case simply  $\mathcal{R}_B = \mathcal{R}_{\mathbf{Q}_p} \otimes_{\mathbf{Q}_p} B$ . We will need to refer to

Fontaine's  $2\pi i$ -element  $t$ , the objects of character type  $\mathcal{R}_B(\delta)$  (where  $\delta: \mathbf{Q}_p^\times \rightarrow B^\times$  is a continuous character), and their Herr cohomology; see [KPX] for details. We denote by  $x, |x|: \mathbf{Q}_p^\times \rightarrow \mathbf{Q}_p^\times$  the identity map and  $p$ -adic absolute value, respectively, when we wish to emphasize them as continuous characters of  $\mathbf{Q}_p^\times$ . Then, for example,  $\mathcal{R}_{\mathbf{Q}_p}(x) \cong t\mathcal{R}_{\mathbf{Q}_p} \subset \mathcal{R}_{\mathbf{Q}_p}$  and  $\mathcal{R}_{\mathbf{Q}_p}(x \cdot |x|) \cong \mathbf{D}_{\text{rig}}(\mathbf{Q}_p(1))$ .

### §1.3 Extensions of characters and their $\mathcal{L}$ -invariants

In this subsection  $E/\mathbf{Q}_p$  is a finite field extension, and we apply system of §1.1 to the data  $X = H^1(E(1))$ ,  $X^* = H^1(E)$ ,  $\lambda^* = \log_p$ ,  $\mu^* = \text{ord}_p$ . Fix also continuous characters  $\delta_0, \eta_0: \mathbf{Q}_p^\times \rightarrow E^\times$ , and a  $(\varphi, \Gamma)$ -module  $D_0$  over  $\mathcal{R}_E$  sitting in a short exact sequence

$$0 \rightarrow \mathcal{R}_E(\eta_0) \rightarrow D_0 \rightarrow \mathcal{R}_E(\delta_0) \rightarrow 0.$$

This sequence defines an extension class  $[D_0]$  in

$$\text{Ext}^1(\mathcal{R}_E(\delta_0), \mathcal{R}_E(\eta_0)) = \text{Ext}^1(\mathcal{R}_E, \mathcal{R}_E(\eta_0\delta_0^{-1})) = H^1(\mathcal{R}_E(\eta_0\delta_0^{-1})),$$

and knowledge of the span  $E[D_0] \subseteq H^1(\mathcal{R}_E(\eta_0\delta_0^{-1}))$  is equivalent to knowledge of the isomorphism class of the  $(\varphi, \Gamma)$ -module  $D_0$  over  $\mathcal{R}_E$ . We restate the well-known classification of  $D_0$  via (dual)  $\mathcal{L}$ -invariants.

*First case:*  $\eta_0$  is not of the form  $(x \cdot |x|)x^k\delta_0$  or  $x^{-k}\delta_0$  for any integer  $k \geq 0$ . Then one has  $\dim_E H^1(\mathcal{R}_E(\eta_0\delta_0^{-1})) = 1$ , so the nonsplit  $D_0$  are all isomorphic.

*Second case:*  $\eta_0 = (x \cdot |x|)x^k\delta_0$  for some integer  $k \geq 0$ . Thus  $\mathcal{R}_E(\eta_0) \cong t^k\mathcal{R}_E(\delta_0)(1)$ , and one has

$$H^1(\mathcal{R}_E(\eta_0\delta_0^{-1})) \cong H^1(t^k\mathcal{R}_E(1)) \xrightarrow{\sim} H^1(\mathcal{R}_E(1)) = H^1(E(1)) = \mathbf{Q}_p^{\times, \wedge} \otimes_{\mathbf{Z}_p} E.$$

We write  $q_{D_0}$  for the image of  $[D_0]$  under this identification. If  $q_{D_0} \neq 0$ , then knowledge of  $D_0$  up to isomorphism is equivalent to knowledge of its (*Fontaine–Mazur*)  $\mathcal{L}$ -invariant  $\mathcal{L}(D_0) = \mathcal{L}(q_{D_0}) = (\log_p q_{D_0} : \text{ord}_p q_{D_0})$ .

Such  $D_0$  is semistable up to twist, and is moreover crystalline up to twist if and only if  $\text{ord}_p q_{D_0} = 0$ , that is, either  $q_{D_0} = 0$  or both  $q_{D_0} \neq 0$  and  $\mathcal{L}(D_0) = \infty$ . Conversely, whenever  $D_0$  has rank two and is up to twist noncrystalline semistable, we are in the above situation for uniquely determined  $\delta_0$  and  $k$ , and  $\text{ord}_p q_{D_0} \neq 0$  so that  $\mathcal{L}(D_0)$  is defined and  $\mathcal{L}(D_0) \neq \infty$ . Such  $D_0$  are not isomorphic to those arising in any other of these three cases. In the noncrystalline semistable case, a computation in Fontaine's theory shows that if  $e \in \mathbf{D}_{\text{st}}(D_0(\delta_0^{-1}))$  is a  $\varphi$ -eigenvector mapping to a basis of  $\mathbf{D}_{\text{crys}}(\mathcal{R}_E)$ , then  $\mathcal{L}(D_0)$  is the slope of the Hodge filtration on  $\mathbf{D}_{\text{dR}}(D_0(\delta_0^{-1})) = \mathbf{D}_{\text{st}}(D_0(\delta_0^{-1}))$  relative to the basis  $e, N(e)$ . (There exist plenty rank two crystalline  $D$  that are not extensions of some  $\mathcal{R}_E(\delta_0)$  by  $t^k\mathcal{R}_E(\delta_0)(1)$ .)

*Third case:*  $\eta_0 = x^{-k}\delta_0$  for some integer  $k \geq 0$ . Thus  $\mathcal{R}_E(\eta_0) \cong t^{-k}\mathcal{R}_E(\delta_0)$ , and one has

$$H^1(\mathcal{R}_E(\eta_0\delta_0^{-1})) \cong H^1(t^{-k}\mathcal{R}_E) \xleftarrow{\sim} H^1(\mathcal{R}_E) = H^1(E) = \text{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p^{\times, \wedge}, E).$$

We write  $q_{D_0}^*$  for the image of  $[D_0]$  under this identification. If  $q_{D_0}^* \neq 0$ , then knowledge of  $D_0$  up to isomorphism is equivalent to knowledge of its (*Hodge–Tate*) dual  $\mathcal{L}$ -invariant  $\mathcal{L}^*(D_0) = \mathcal{L}^*(q_{D_0}^*) = (-\langle \text{ord}_p, q_{D_0}^* \rangle : \langle \log_p, q_{D_0}^* \rangle)$ .

For example, all rank one objects are of the form  $\mathcal{R}_E(\delta_0)$  for uniquely determined  $\delta_0$ , so the case where  $k = 0$  is none other than the situation where  $D_0$  is an extension of a general rank one object by itself. This extension is nonsplit if and only if  $q_{D_0}^* \neq 0$ , in which case  $D_0$  is Hodge–Tate up to twist if and only if  $\mathcal{L}^*(D_0) = \infty$ .

## §2 $\mathcal{L}$ -invariants of specializations of families

Throughout this section, we fix a finite extension  $E/\mathbf{Q}_p$ , and a first-order deformation  $(B, \mathfrak{m})$  of  $E$ , to be defined immediately below.

### §2.1 First order deformations

By a *first-order deformation of  $E$* , we mean an Artinian local  $E$ -algebra  $(B, \mathfrak{m})$  with  $B/\mathfrak{m} = E$  and  $\mathfrak{m}^2 = 0$ , and by a morphism of these we mean a local  $E$ -algebra map. One has  $B = E \oplus \mathfrak{m}$  as  $E$ -vector spaces, and  $B^\times = E^\times \times (1 + \mathfrak{m})$ . Note that the Kähler derivative  $d: B = E \oplus \mathfrak{m} \rightarrow \Omega_{B/E}$  is zero on the first factor and an isomorphism  $\mathfrak{m} \cong \Omega_{B/E}$  on the second, and employing  $d$  value-by-value with respect to  $\mathbf{Q}_p^{\times, \wedge}$  we deduce an identification  $d: H^1(E) \otimes_E \mathfrak{m} = \text{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p^{\times, \wedge}, \mathfrak{m}) \cong \text{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p^{\times, \wedge}, \Omega_{B/E})$ .

We may view  $\langle \log_p, \cdot \rangle$ ,  $\langle \text{ord}_p, \cdot \rangle$  and  $q \in \mathbf{Q}_p^{\times, \wedge}$  as maps  $H^1(E) \rightarrow E$ , and thus also as maps  $H^1(E) \otimes_E \mathfrak{m} \rightarrow \mathfrak{m}$ . For  $c \in H^1(E) \otimes_E \mathfrak{m}$  we have  $\langle \log_p, c \rangle, \langle \text{ord}_p, c \rangle, c(q) \in \mathfrak{m}$ , and

$$\log_p \otimes \langle \log_p, c \rangle + \text{ord}_p \otimes \langle \text{ord}_p, c \rangle = c.$$

Evaluating the preceding equation at  $q = p$ , we find that  $\langle \text{ord}_p, c \rangle = c(p)$ . On the other hand, evaluating at any nonidentity  $\gamma_0 \in 1 + 2p\mathbf{Z}_p$ , we find that  $\langle \log_p, c \rangle = \frac{c(\gamma_0)}{\log_p \gamma_0}$ .

Let  $c \in H^1(E) \otimes_E \mathfrak{m}$  be given. Writing  $\mathfrak{m}^* = \text{Hom}_E(\mathfrak{m}, E)$ , it is easy to see that the following conditions on  $c$  are equivalent:

- There exists nonzero  $q \in H^1(E(1))$  such that  $c \in H^1(E)^{q=0} \otimes_E \mathfrak{m}$ .
- There exists nonzero  $q \in H^1(E(1))$  such that  $(q \otimes 1)(c) = 0$ .
- One of  $\langle \log_p, c \rangle, \langle \text{ord}_p, c \rangle \in \mathfrak{m}$  is an  $E$ -multiple of the other.
- There exist  $h \in H^1(E)$  and  $m \in \mathfrak{m}$  such that  $c = h \otimes m$ .
- The  $E$ -subspace  $L_c = \{(1 \otimes v)(c) \mid v \in \mathfrak{m}^*\} \subseteq H^1(E)$  satisfies  $\dim_E L_c \leq 1$ .

If these conditions are satisfied (for example, whenever  $\dim_E \Omega_{B/E} = 1$ ) and moreover  $c \neq 0$ , we call  $c$  a *pure tensor*. In this case  $q, h$ , and  $m$  are uniquely determined up to nonzero  $E^\times$ -multiples, one has  $H^1(E)^{q=0} = L_c = Eh$ , and the lines  $Eq \subseteq H^1(E(1))$  and  $Eh \subseteq H^1(E)$  are an orthogonal pair. The orthogonality shows that  $\mathcal{L}(q) = \mathcal{L}^*(h)$ , and this common quantity is the constant of proportionality (with slight abuse of notation)  $(-\langle \text{ord}_p, c \rangle : \langle \log_p, c \rangle) \in \mathbf{P}^1(E)$ ; we denote it by  $\mathcal{L}^*(c)$ . By definition, if  $\mathcal{L}^*(c) \neq \infty$  one has

$$\langle \text{ord}_p, c \rangle = -\mathcal{L}^*(c) \langle \log_p, c \rangle. \quad (1)$$

### §2.2 Characters valued in first-order deformations

A continuous character  $\delta: \mathbf{Q}_p^\times \rightarrow B^\times$  can be written uniquely in the form  $\delta = \delta_0 \cdot (1 + \delta_1)$  where  $\delta_0: \mathbf{Q}_p^\times \rightarrow E^\times$  and  $\delta_1: \mathbf{Q}_p^{\times, \wedge} \rightarrow \mathfrak{m}$  are continuous homomorphisms. We use subscripts to denote the formation of these components; one has  $(\delta\eta)_0 = \delta_0\eta_0$  and  $(\delta\eta)_1 = \delta_1 + \eta_1$ , and for a morphism  $f: B \rightarrow B'$  one has  $(f \circ \delta)_0 = f \circ \delta_0$  and  $(f \circ \delta)_1 = f \circ \delta_1$ . One has the short exact sequence

$$0 \rightarrow \mathcal{R}_E(\delta_0) \otimes_E \mathfrak{m} \rightarrow \mathcal{R}_B(\delta) \rightarrow \mathcal{R}_E(\delta_0) \rightarrow 0,$$

and the corresponding extension class in

$$\mathrm{Ext}^1(\mathcal{R}_E(\delta_0), \mathcal{R}_E(\delta_0) \otimes_E \mathfrak{m}) = H^1(\mathcal{R}_E \otimes_E \mathfrak{m}) = H^1(E) \otimes_E \mathfrak{m} \stackrel{q^*}{=} \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p^{\times, \wedge}, \mathfrak{m})$$

is computed by  $\delta_1$ , which we often view via  $(q^*)^{-1}$  as an element of  $H^1(E) \otimes_E \mathfrak{m}$ . The sequence is thus nonsplit if and only if  $\delta_1 \neq 0$ , if and only if  $\mathcal{R}_B(\delta)$  is not isomorphic to  $\mathcal{R}_E(\delta_0) \otimes_E B$ .

We reinterpret the quantities of §2.1 in differential language when  $c = \delta_1$  for a continuous character  $\delta: \mathbf{Q}_p^\times \rightarrow B^\times$ , assuming that  $\delta_1$  is a pure tensor. We compute the image of  $\delta_1$  under the identification  $d: H^1(E) \otimes_E \mathfrak{m} \cong \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p^{\times, \wedge}, \Omega_{B/E})$ , value-by-value with respect to  $\mathbf{Q}_p^{\times, \wedge}$ , to be

$$d(\delta_1) = \frac{\delta_0 d(\delta_1)}{\delta_0} = \frac{d(\delta_0 + \delta_0 \delta_1)}{\delta_0 + \delta_0 \delta_1} = \frac{d(\delta_0(1 + \delta_1))}{\delta_0(1 + \delta_1)} = d \log(\delta),$$

where the second step is because  $\delta_0$  (resp.  $\delta_1$ ) takes values in  $E$  (resp.  $\mathfrak{m}$ ). In particular, evaluating both sides at  $p$ , we find that

$$d\langle \mathrm{ord}_p, \delta_1 \rangle = d(\delta_1(p)) = d \log(\delta(p)).$$

Next we compute the image of  $d\langle \log_p, \delta_1 \rangle \in \mathfrak{m}$  under the identification to be

$$d\langle \log_p, \delta_1 \rangle = \frac{d\delta_1(\gamma_0)}{\log_p \gamma_0} = \frac{d \log(\delta(\gamma_0))}{\log_p \gamma_0} = -d \mathrm{wt}(\delta),$$

where  $\mathrm{wt}(\delta) \in B$  is the *weight* of  $\delta$ , defined as the value of  $-\frac{\log \delta(\gamma_0)}{\log_p \gamma_0}$  for  $\gamma_0 \in 1 + 2p\mathbf{Z}_p$  nonidentity and sufficiently close to the identity. This invariant is normalized so that  $\mathrm{wt}(\delta)$  agrees with the Sen weight of  $\mathcal{R}_B(\delta)$ , where we consider the Sen weight of  $\mathbf{Q}_p(1)$  to be  $-1$ . Substituting these calculations of  $d\langle \mathrm{ord}_p, \delta_1 \rangle$  and  $d\langle \log_p, \delta_1 \rangle$  into the equation (1) above, we find that if  $(\delta_1$  is a pure tensor and)  $\mathcal{L}^*(\delta_1) \neq \infty$  then

$$d \log(\delta(p)) = \mathcal{L}^*(\delta_1) \cdot d \mathrm{wt}(\delta) \text{ in } \Omega_{B/E}. \quad (2)$$

### §2.3 Extensions of characters over first-order deformations

We suppose given continuous characters  $\delta, \eta: \mathbf{Q}_p^\times \rightarrow B^\times$  and a short exact sequence

$$0 \rightarrow \mathcal{R}_B(\eta) \rightarrow D \rightarrow \mathcal{R}_B(\delta) \rightarrow 0,$$

and set  $D_0 = D \otimes_B E$ . We take Herr cohomology of the short exact sequence

$$0 \rightarrow \mathcal{R}_E((\eta\delta^{-1})_0) \otimes_E \mathfrak{m} \rightarrow \mathcal{R}_B(\eta\delta^{-1}) \rightarrow \mathcal{R}_E((\eta\delta^{-1})_0) \rightarrow 0$$

to obtain the exact sequence

$$H^1(\mathcal{R}_B(\eta\delta^{-1})) \xrightarrow{\alpha} H^1(\mathcal{R}_E((\eta\delta^{-1})_0)) \xrightarrow{\partial} H^2(\mathcal{R}_E((\eta\delta^{-1})_0)) \otimes_E \mathfrak{m}, \quad (3)$$

in which the reduction map  $\alpha$  sends the class of  $D$  to  $[D_0]$ , and the connecting map  $\partial$  is given (up to sign) by cup product with  $(\eta\delta^{-1})_1 \in H^1(E) \otimes_E \mathfrak{m}$ . In particular,  $(\eta\delta^{-1})_1 \cup [D_0] = 0$ .

Now we treat separately the three cases of §1.3 applied to  $D_0$ .

*Second case:* The Tate duality isomorphism  $H^2(\mathcal{R}_E((\eta\delta^{-1})_0)) \otimes_E \mathfrak{m} \cong \mathfrak{m}$  corresponds  $(\eta\delta^{-1})_1 \cup [D_0]$  with  $(\eta\delta^{-1})_1(q_{D_0})$  (up to sign), so that  $(\eta\delta^{-1})_1(q_{D_0}) = 0$ .

*Proof of the theorem.* By replacing  $X$  by its first-order infinitesimal neighborhood at  $P$  and  $E$  by the residue field at  $P$ , we may assume  $X$  is the spectrum of a first-order deformation of  $E$ . Recall that  $D_0$  is noncrystalline semistable up to twist, so that according to §1.3 we are in the situation immediately preceding this proof. In particular, one has  $\text{ord}_p(q_{D_0}) \neq 0$  and so  $q_{D_0} \neq 0$ , and also that  $\mathcal{L}(D_0)$  is defined and  $\mathcal{L}(D_0) \neq \infty$ .

If  $(\eta\delta^{-1})_1 = 0$  then  $d \log(\eta\delta^{-1}(p)) = d \text{wt}(\eta\delta^{-1}) = 0$ , and we are done.

If  $(\eta\delta^{-1})_1 \neq 0$  then the identity  $(\eta\delta^{-1})_1(q_{D_0}) = 0$  implies that  $(\eta\delta^{-1})_1$  is a pure tensor, and by orthogonality  $\mathcal{L}(D_0) = \mathcal{L}^*((\eta\delta^{-1})_1)$ . The theorem now follows from (2) applied to the character  $\eta\delta^{-1}$ .  $\square$

The two main ingredients in the proof of the theorem, namely the computation (2) and the equation  $(\eta\delta^{-1})_1(q_{D_0}) = 0$  coming from the long exact sequence on local Galois cohomology, are also key steps in the proof of [GS, (0.15)].

*First and third cases:* Now one has  $H^2(\mathcal{R}_E((\eta\delta^{-1})_0)) = 0$ , so the exactness of (3) implies the reduction map  $\alpha$  is surjective.

In the first case, we see that the unique split and nonsplit possibilities for  $D_0$  may occur.

In the third case,  $[D_0]$  is identified to  $q_{D_0}^* \in H^1(E)$ . The surjectivity of the reduction map  $\alpha$  shows that the two subspaces  $Eq_{D_0}^* \subseteq H^1(E)$  and  $E(\eta\delta^{-1})_1 \subseteq H^1(E) \otimes_E \mathfrak{m}$  can be arbitrary of dimension at most one. Assuming that  $[D_0] \neq 0$ , so the dual  $\mathcal{L}$ -invariant  $\mathcal{L}^*(D_0)$  is defined, this independence of  $q_{D_0}^*$  and  $(\eta\delta^{-1})_1$  shows that no formula for  $\mathcal{L}^*(D_0)$  purely in terms of  $\delta, \eta$  is possible without further hypotheses.

## References

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